Plasma Physics

Waves in Plasma

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Maxwell Equations

Charge and current densities:

$$\rho = \sum_\alpha n_\alpha q_\alpha$$

$$\mathbf{j} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha$$

Maxwell's equations:

$$\begin{aligned}
\varepsilon_0 \nabla \cdot \mathbf{E} &= \sum_\alpha n_\alpha q_\alpha = \rho \\
\nabla \times \mathbf{E} &= \dot{\mathbf{B}} \\
\nabla \cdot \mathbf{B} &= 0 \\
\mu_0^{-1} \nabla \times \mathbf{B} &= \varepsilon_0 \dot{\mathbf{E}} + \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha \\
&= \varepsilon_0 \dot{\mathbf{E}} + \mathbf{j}
\end{aligned}$$

Waves Equation in Plasma

Waves equation in the Fourier space

Following the usual procedure, $\nabla\times$ of II is taken and IV is used:

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\left[\frac{1}{\varepsilon_0 c^2}\mathbf{j} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right]$$

The Fourier transform of the equations is taken, where for

$$\mathbf{E}_{\mathbf{k},\omega} = \mathbf{E_0} \exp\left[\mathrm{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]$$

it holds

$$\left\{ \begin{array}{rcl} \mathcal{F}\,\nabla\times & = & -\mathrm{i}\mathbf{k}\times \\ \mathcal{F}\,\partial_t & = & -\mathrm{i}\omega \end{array} \right.$$

for which

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} = -\mathrm{i}\omega\left[\frac{1}{c^2\varepsilon_0}\mathbf{j} + \frac{\mathrm{i}\omega}{c^2}\mathbf{E}\right]$$

Waves Equation in Plasma

Conductivity and Permittivity tensors

We define the conductivity tensor as

$$\mathbf{j} = \underline{\underline{\sigma}} \cdot \mathbf{E}$$

then

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} = \frac{\omega^2}{c^2} \left[\underline{\underline{\mathbb{1}}} - \frac{\mathrm{i}}{\varepsilon_0 \omega} \underline{\underline{\sigma}} \right] \mathbf{E}$$

From this expression it is defined the tensor form of the permittivity:

$$\underline{\underline{\varepsilon}} \equiv \left[\underline{\underline{\mathbb{1}}} - \frac{\mathrm{i}}{\varepsilon_0 \omega} \underline{\underline{\sigma}} \right]$$

The wave equation for a conducting fluid can then be written as

$$\left[\left(\underline{\underline{\mathbb{1}}} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) \frac{c^2 k^2}{\omega^2} - \underline{\underline{\varepsilon}} \right] \mathbf{E} = 0$$

Waves Equation in Plasma

General considerations

The equation in the form

$$\left[\left(\underline{\underline{\mathbb{1}}} - \frac{\mathbf{k}\mathbf{k}}{k^2}\right)\frac{c^2k^2}{\omega^2} - \underline{\underline{\varepsilon}}\right]\mathbf{E} = 0$$

is the most general expression we can give. It's worth to underline that:

- it represents a set of three equations (non banal solutions for $\det\left[\cdots\right] = 0$);
- the physics is yet to be set via the conductivity tensor $\underline{\underline{\sigma}}$

Plasma conductivity in the cold plasma approximation

We wish to derive $\underline{\underline{\sigma}}$ from the fluid equations. At the first order we have:

$$\frac{\partial n}{\partial t} + \nabla \cdot (\mathbf{u} n) = 0$$

$$mn \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = nq \left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) - \nabla \cdot \underline{\underline{\mathbf{P}}}$$

where $\underline{\underline{\mathbf{P}}} = mn \langle \mathbf{w}\mathbf{w} \rangle$ is the highest order momentum. In the "cold plasma approximation" we neglect the thermal movement of the particles:

$$\mathbf{w} \to 0 \;\Rightarrow \underline{\underline{\mathbf{P}}} = 0$$

By solving the equations (first order perturbative) one gets

$$\mathbf{u}_\alpha = \mathrm{i} \frac{q_\alpha}{\omega m_\alpha} \mathbf{E}$$

$$\mathbf{j}_\alpha = n_\alpha q_\alpha \mathbf{u}_\alpha = \mathrm{i} \frac{n_\alpha q_\alpha^2}{\omega m_\alpha} \mathbf{E}$$

Cold plasma permittivity

From the definitions:

$$\underline{\underline{\varepsilon}} = \underline{\underline{\mathbb{1}}} - \frac{\mathrm{i}}{\varepsilon_0 \omega} \underline{\underline{\sigma}}$$
$$= \left[1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} \right] \underline{\underline{\mathbb{1}}}$$

According to this result cold plasma permittivity is a scalar:

$$\underline{\underline{\varepsilon}}_{\,cold} = \varepsilon_{cold} \, \underline{\underline{\mathbb{1}}}$$

Setting $\widehat{\mathbf{k}} \parallel \widehat{\mathbf{z}}$ the wave equation reduces to:

$$\begin{bmatrix} \frac{c^2 k^2}{\omega^2} - \varepsilon & 0 & 0 \\ 0 & \frac{c^2 k^2}{\omega^2} - \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \mathbf{E} = 0$$

Two kinds of solutions are to be explored:

- $E_z \neq 0$: longitudinal waves
- $E_z = 0$: transverse waves

Longitudinal waves

For longitudinal waves ($E_z \neq 0$) the permittivity voids:

$$\varepsilon = 0 \longrightarrow \omega^2 = \omega_{p\alpha}$$

This represents a purely electrostatic space charge wave
In a cold plasma there is no spatial dispersion ($\mathbf{j}\left(\mathbf{x}\right)$ is function of only local value of $\mathbf{E}$)

Since ω is independent of $\mathbf{k}$:

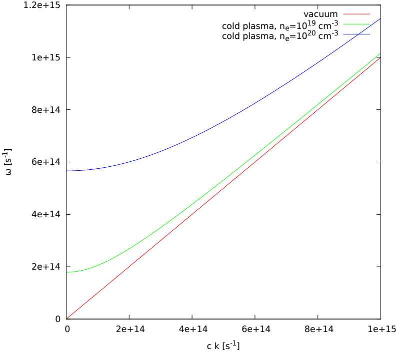
$$v_g = \frac{\partial \omega}{\partial k} = 0$$

hence the wave does not propagate.

Waves in Cold Plasma

Transverse waves

For waves with $E_z = 0$ the existing solution is for $\underline{\underline{\varepsilon}} = \frac{c^2 k^2}{\omega^2}$ hence



$$\frac{c^2 k^2}{\omega^2} = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}$$

In most cases simplified to

$$\frac{c^2 k^2}{\omega^2} \simeq 1 - \frac{\omega_{pe}^2}{\omega^2}$$

In conclusion:

- $\frac{c^2 k^2}{\omega^2} = \frac{c^2}{v_{phase}^2} < 1 \Rightarrow v_{phase} > c$
- cutoff frequency: $\omega < \omega_{pe}$ waves cannot propagate