# Chapter 5

# Longitudinal waves in plasma

This chapter is dedicated to the properties of the electron oscillation in a plasma and to define how such oscillation can become a wave, i.e. how it can propagate.

Apart from the intrinsic importance of electron longitudinal waves, the **Langmuir waves**, we present how two different derivations can lead to different results and how the kinetic treatment of the problem make evident properties (the **Landau damping**) that could not be even foreseen from a fluid representation of the plasma.

This evidence stresses once again the importance of the concept of **thermal equilibrium** in a plasma.

## 5.1 The plasma oscillation

The existence of a proper oscillation frequency of the electron component in a plasma can be derived from general considerations. Let's suppose we have neutral plasma with $Z = 1$ ions of infinite mass. The plasma is isotropic and initially at rest.

In order to describe the dynamics of the electron component we use the Gauss law (5.1) and the continuity equation (5.2). In order to introduce the reaction of the electron component to perturbations in the electric field we use a fluid generalization of the Lorenz force (5.3), limited to the electric component[1]

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho \tag{5.1}$$

$$\frac{\partial n_e}{\partial t} + \nabla \left( \mathbf{u} n_e \right) = 0 \tag{5.2}$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{e}{m_e} \mathbf{E} \tag{5.3}$$

[1]This relationship between fluid velocity and electric field is introduced here from purely general considerations, which are proved correct, although limited in scope, by the derivation of fluid equations from the kinetic theory.

The perturbative calculation is carried out at the first order, setting

$$\begin{aligned} n_e &= n_0 + n_1 \\ \mathbf{u} &= \mathbf{u_0} + \mathbf{u_1} \\ \mathbf{E} &= \mathbf{E_0} + \mathbf{E_1} \end{aligned} \tag{5.4}$$

where

$$\nabla n_0 = \mathbf{u_0} = \mathbf{E_0} = 0 \tag{5.5}$$

From simple substitutions (left to the reader) one easily gets to

$$\left( \frac{\partial^2}{\partial t^2} + \frac{n_0 e^2}{m_e \varepsilon_0} \right) n_1 = 0 \tag{5.6}$$

which is an harmonic oscillator in $n_1 \equiv n_1(\mathbf{x}, t)$. Taking the solution $n_1()$ in its most general form

$$n_1(\mathbf{x}, t) = \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{5.7}$$

we finally have

$$\begin{cases} \omega = \left( \dfrac{n_0 e^2}{m_e \varepsilon_0} \right)^{1/2} \\ \\ \dfrac{\partial \mathbf{k}}{\partial \omega} = 0 \end{cases} \tag{5.8}$$

The first equation of (5.8) represents the self-oscillating frequency of the electron component in the plasma. The second one shows that no group velocity exists for the oscillation we found. This indicates that even if an oscillating frequency can be defined, no wave is produced, and that the oscillation remains local.

## 5.2 The fluid treatment: Langmuir waves

The result in (5.8) can be refined by a more careful fluid treatment. This is obtained by starting from the full equation of the **first order momentum**:

$$mn \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla \cdot \underline{\underline{\mathbf{P}}} \tag{5.9}$$

instead of (5.3) in the previous calculation.

### 5.2.1 On the cold plasma approximation

The **stress tensor** $\underline{\underline{\mathbf{P}}}$ in (5.9) was defined as

$$\underline{\underline{\mathbf{P}}} = mn \langle \mathbf{w}\mathbf{w} \rangle \tag{5.10}$$

where $\mathbf{w} \equiv \mathbf{v} - \mathbf{u}$ is the thermal component of the velocity, $\mathbf{u}$ being the fluid one. Its tensorial nature expresses the existence of all the possible correlations between the different components of the thermal velocity. Diagonal terms, $w_i^2$, represent pressure, while off-diagonal ones, $(w_i\, w_j)_{i \neq j}$ represent shear forces.

In the case of an isotropic plasma, thermal velocities don't have a preferential direction, hence $\mathbf{w} = w\mathbb{1}$; moreover, supposing that the plasma is non-viscous, off-diagonal terms are void. Under these assumptions, the stress tensor become a scalar, and can be rewritten as

$$\underline{\underline{\mathbf{P}}} = mnw^2 \underline{\underline{\mathbb{1}}} \equiv p \underline{\underline{\mathbb{1}}} \tag{5.11}$$

where $p$ is now the pressure in the usual meaning of the term.

In a plasma where a thermal equilibrium has been reached and where a temperature can be defined the term $w^2$ can be expressed from the most probable velocity of particles in the plasma.

In the specific case of a plasma at Maxwellian equilibrium, $\langle w^2 \rangle = 3k_B T/m$ which gives

$$p = 3nk_B T \tag{5.12}$$

the scalar pressure in a **isotropic**, **non-viscous plasma**.

Equation (5.3) then corresponds to the first order fluid equation (5.9) [2] in the limit $T = 0$, which is termed **cold plasma approximation**.

### 5.2.2 Longitudinal waves in hot plasma

Using first order momentum equation in a plasma at Maxwellian equilibrium with temperature $T$ instead of (5.3) gives now[3]

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho \tag{5.13}$$

$$\frac{\partial n_e}{\partial t} + \nabla \left( \mathbf{u} n_e \right) = 0 \tag{5.14}$$

$$mn \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -en\mathbf{E} - 3k_B T \nabla n \tag{5.15}$$

The searched solution is in the form (5.7); if we limit the calculation to a longitudinal wave along $\hat{x}$ operators get simplified as follows:

$$\begin{aligned} \mathbf{k} \cdot \mathbf{x} &= k_x x \\ \partial_t &= -\mathrm{i}\omega \\ \nabla &= \mathrm{i}k_x \hat{x}. \end{aligned}$$

[2]This is true when solving the problem for a purely longitudinal wave, $\mathbf{u} \times \mathbf{B} = 0$; otherwise the magnetic term would have been added to equation (5.3)

[3]Notice that the $\mathbf{u} \times \nabla$ term of the convective derivative is void in a first order linearization of the problem, being $\mathbf{u}_0 = 0$.

By substitution, and remembering the plasma frequency definition, one finally gets:

$$\begin{aligned}\omega^2 &= \omega_{pe}^2 + \frac{3k_BT}{m}k_x^2 \\ &= \omega_{pe}^2 + \frac{3}{2}k_x^2 v_{th}^2 \end{aligned} \tag{5.16}$$

where $v_{th}^2 = 2k_BT/m$ is the most probable velocity in a Maxwellian plasma at thermal equilibrium with temperature $T$.

The frequency in (5.16) now depends on $k$ (via $T \neq 0$), hence a finite group velocity exists:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{3}{2}\frac{k}{\omega}v_{th}^2 \tag{5.17}$$

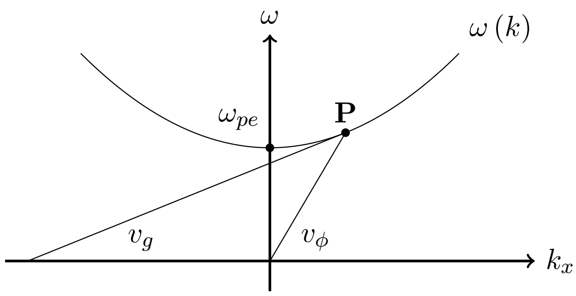


Figure 5.1: Dispersion law for electron plasma waves

In conclusion we found that a longitudinal electron wave can propagate in a plasma, provided pressure contributions are taken in account. This is in principle no different from what is obtained for sound propagation in a gas.
In Fig.5.1, a graphical representation of the plasma dispersion relation (5.16) is provided. The plasma electron frequency act as a *cutoff* frequency: no wave at $\omega < \omega_{pe}$ can propagate. This fact can also be seen as the electrons in the plasma being able to effectively replenish any electrostatic perturbation with a period that is *slower* than the plasma oscillation period.
For waves with $\omega > \omega_{pe}$, $k > 0$, hence $v_\phi \neq 0$ and $v_g \neq 0$. Given a point $P(\omega, k)$ on the dispersion law, the group velocity $v_g$ can be visualised via the tangent to the parabola in $P$.

## 5.3 The kinetic treatment: Landau damping

As a further refinement to the theory of longitudinal waves in a plasma we want to get rid of the fluid representation and to observe what the effect could be of electron oscillations in a kinetic representation.
The question we want to answer is whether the electron velocity distribution has an impact to the dynamics of the longitudinal wave propagation. The interest in making this additional step can be guessed from equation (5.16), which depends on the electron thermal velocity (which we averaged under the hypothesis of thermal equilibrium. . . ).

Let's write the distribution function as

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_1(\mathbf{x}, \mathbf{v}, t) \tag{5.18}$$

where $f_0\left(\right)$ is the initial unperturbed, isotropic, homogeneous function. From Vlasov equation on $f\left(\right)$ (modifying (5.4) accordingly and noticing that now $\mathbf{v}$ is independent, hence not linearized) we get

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \frac{q}{m}\mathbf{E_1} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{5.19}$$

We follow the same procedure as in section 5.2.2 for solutions of the kind

$$f_1\left(\right) \propto \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{5.20}$$

If we limit the solutions to waves along $\hat{x}$, and remembering that $\int f_1 \mathrm{d}^3 v = n_1$ the solution is simplified to the form

$$1 = \frac{\omega_{pe}}{k_x^2} \int \frac{\partial \hat{f}_0/\partial v_x}{v_x - \omega/k_x} \mathrm{d}v_x. \tag{5.21}$$

The integral equation (5.21) is called **Landau integral** and represents a result of fundamental importance in plasma physics. A complete treatment of this problem and of its consequences goes well beyond the scope of this course. In the following sections a simplified solution will be presented and discussed.

### 5.3.1 Simplified solution of the Landau problem

The integral in (5.21) has a singularity for $v_x = \omega/k_x$. This falls exactly when the particle velocity equals the wave phase velocity, which indicates that problems might arise for particles that propagate **with** the wave.

In the most general cases the imaginary part of the phase velocity Im $(\omega/\mathrm{k}) \neq 0$, meaning that some processes do usually amplify the wave (Im $(\omega/\mathrm{k}) > 0$) or damp it (Im $(\omega/\mathrm{k}) < 0$).

From the mathematical point of view, Im $(\omega/\mathrm{k}) \neq 0$ implies that the denominator of the integrand never voids, so the integration on the real axis would always give a finite result. It has however been proved that the residue due to the singularity in the complex plane is not negligible and that it must be taken in account for a correct solution of the problem.

The usual way to treat such a case would have been via the *residue theorem* (Fig. 5.2):

$$\int_{C_1} G \mathrm{d}v + \int_{C_2} G \mathrm{d}v = 2\pi \mathrm{i} R\left(\omega/k\right) \tag{5.22}$$

If one can prove that the path integral along $C_2$ voids at infinity, the result is obtained from the real axis integration plus the residue $R\left(\omega/k\right)$.

This approach cannot work, though, for in the case of a Maxwellian distribution

$$\mathrm{e}^{-v^2/v_{th}^2} \xrightarrow[v \to \pm \mathrm{i}\infty]{} \infty$$

The integration path prescribed by Landau works in the approximation of small Im $(\omega/\mathrm{k})$; in this case, Fig.5.3, the integration can be carried out on Re (v) plus $2\pi\mathrm{i}$ times half of the residue around $\omega/k$.

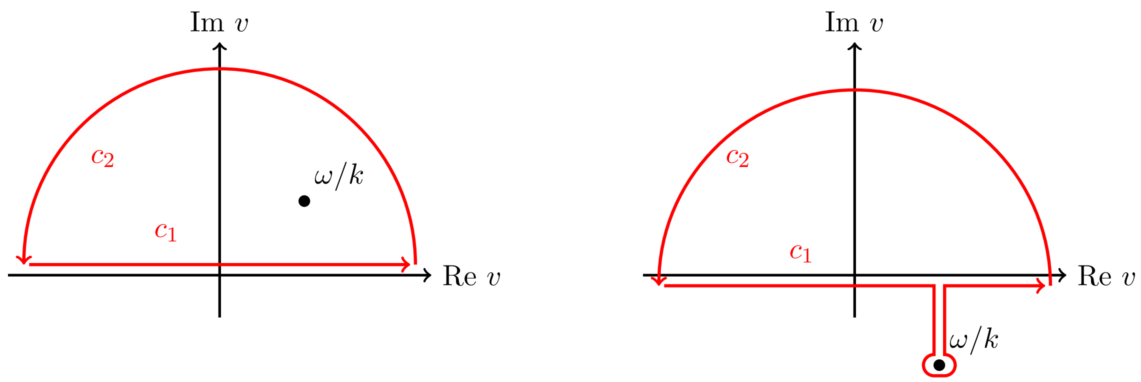


Figure 5.2: Integration contours for the Landau problem for Im $\omega > 0$ and for Im $\omega < 0$

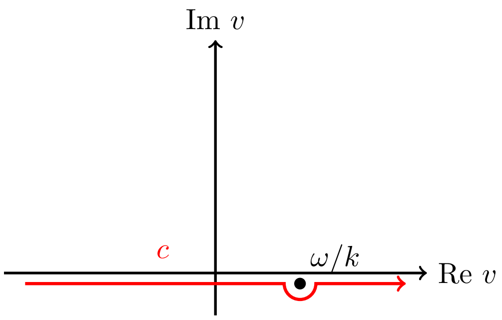


Figure 5.3: Integration contour for small values of Im $\omega$

Under this assumption, integral (5.21) becomes:

$$1 = \frac{\omega_{pe}^2}{k^2} \left[ P \int_{-\infty}^{\infty} \frac{\partial \hat{f}_0 / \partial v_x}{v_x - \omega/k_x} \mathrm{d}v_x + \mathrm{i}\pi \frac{\partial \hat{f}_0}{\partial v_x} \bigg|_{v_x = \omega/k} \right], \tag{5.23}$$

where $P\,()$ is the Cauchy principal value of the function.

### 5.3.2 Real part of the Landau integral: electron waves dispersion law

The real component of (5.23) is limited to the principal value term $P\,()$; integrating by parts it gives

$$\int_{-\infty}^{\infty} \frac{\partial \hat{f}_0}{\partial v_x} \frac{\mathrm{d}v_x}{v_x - v_\phi} = \left[ \frac{\hat{f}_0}{v_x - v_\phi} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-\hat{f}_0 \mathrm{d}v_x}{(v_x - v_\phi)^2} \tag{5.24}$$

$$= \int_{-\infty}^{\infty} \frac{\hat{f}_0 \mathrm{d}v_x}{(v_x - v_\phi)^2} \tag{5.25}$$

which is nothing more than an average of $(v_x - v_\phi)^{-2}$ over the initial distribution, hence

$$1 = \frac{\omega_{pe}^2}{k^2} \left\langle (v_x - v_\phi)^{-2} \right\rangle. \tag{5.26}$$

Under the assumption $v_\phi \gg v_{th}$ the averaged quantity in (5.26) can be developed by Taylor expansion before averaging, which leads to

$$\left\langle (v_x - v_\phi)^{-2} \right\rangle \approx v_\phi^{-2} \left( 1 + \frac{3 \langle v^2 \rangle}{v_\phi^2} \right) \tag{5.27}$$

It is left to the reader to verify that if $f_0$ is Maxwellian, (5.26) equals (5.16), provided (5.27).

### 5.3.3 Consequences of the singularity: the Landau damping

The imaginary part of (5.23) is originated in the residue. In order to keep the $\omega$ dependence, the same calculation as in section 5.3.2 is followed. Additional simplification is added by neglecting the thermal correction (the $v_{th}$ terms) to the real part of the dispersion law. Equation (5.23) simplifies to

$$1 = \frac{\omega_{pe}^2}{\omega^2} + \mathrm{i}\pi \frac{\omega_{pe}^2}{k^2} \left. \frac{\partial \hat{f}_0}{\partial v_x} \right|_{v_x = v_\phi} \tag{5.28}$$

and then to

$$\omega \left( 1 - \mathrm{i}\pi \frac{\omega_{pe}^2}{k^2} \left[ \frac{\partial \hat{f}_0}{\partial v_x} \right]_{v_x = v_\phi} \right)^{1/2} = \omega_{pe} \tag{5.29}$$

From the Taylor expansion of the square root, one finally gets

$$\omega = \omega_{pe} \left( 1 + \mathrm{i} \frac{\pi}{2} \frac{\omega_{pe}^2}{k^2} \left[ \frac{\partial \hat{f}_0}{\partial v_x} \right]_{v = v_\phi} \right) \tag{5.30}$$

which is the imaginary part of the dispersion law from the kinetic representation of the electron plasma waves.

The importance of (5.30) lies is the fact that this modification to the plasma wave dispersion relation **could not be predicted from fluid representation**.

The sign of the term $\left[ \partial \hat{f}_0 / \partial v_x \right]_{v_x = v_\phi}$ depends on the slope of the distribution function $f\,()$ at $v_x = v_\phi$. Being $v_{phi} > v_{th}$, in the Maxwellian case the imaginary term is **negative**, hence resulting in a **collisionless damping of the wave**.

The mechanism can be qualitatively described as follows. Electrons that propagate with a velocity comparable (*resonant*) to the wave phase velocity $v_\phi$ will see the wave as a constant or a slowly varying electrostatic potential. In particular electrons slightly slower than the wave are expected to gain energy from it, while electrons that are slightly faster than the wave are expected to deliver energy to it. However, in a Maxwellian distributions, there are more electrons with $v_x < v_\phi$ than electrons with $v_x > v_\phi$. This unbalance has the net effect of flattening the distribution slope draining energy from the wave, hence damping it (Fig. 5.4).

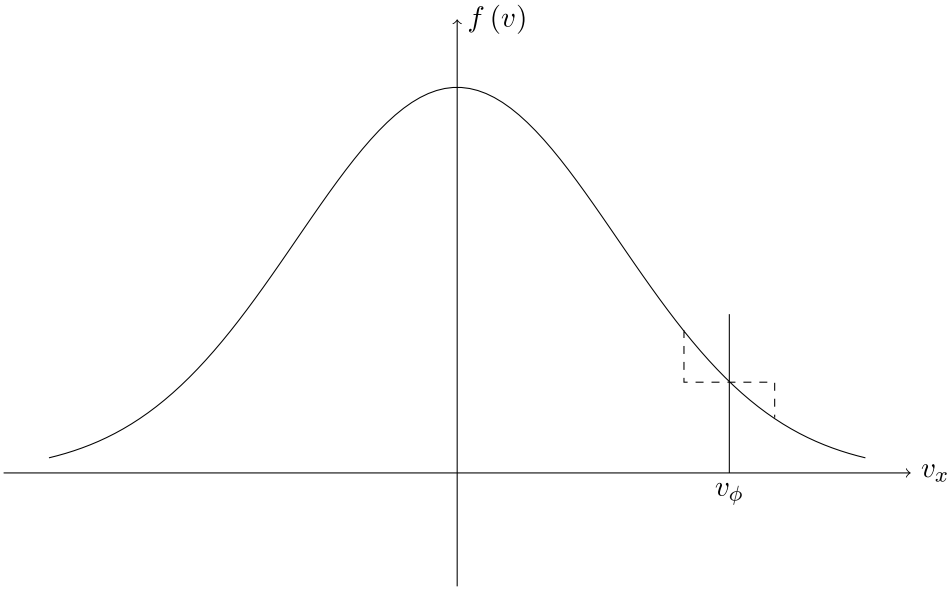


Figure 5.4: Distortion of a Maxwellian distribution in the region $v_x \simeq v_\phi$ caused by Landau damping.