Chapter 3

Collisions

3.1 Elastic collisions

In an elastic collision, like the one from the Coulomb interaction $\mathbf{F} = q_1 q_2 \,/\, \left(4\pi\varepsilon_0 r^2\right) (\mathbf{r}/r)$ between two particles, the kinetic energy and the momentum are conserved:

$$\begin{cases} \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 &= \frac{1}{2}v'^2_1 + \frac{1}{2}v'^2_2 \\ m_1\mathbf{v}_1 + m_2\mathbf{v}_2 &= m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2. \end{cases} \tag{3.1}$$

These equations can be greatly simplified if the problem is treated in the mass center of the system. Given $\mathbf{u}$ the velocity of the center of mass, it holds:

$$\mathbf{u} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \Rightarrow \begin{cases} \mathbf{v}_{10} &= \mathbf{v_1} - \mathbf{u} = \frac{m_2}{m_1 + m_2}\mathbf{v} \\ \mathbf{v}_{20} &= \mathbf{v_2} - \mathbf{u} = \frac{m_1}{m_1 + m_2}\mathbf{v} \end{cases} \tag{3.2}$$

where $\mathbf{v} \equiv \mathbf{v_1} - \mathbf{v_2}$.

The conservation of the momentum ensures that the two particles remain in the plane $\mathbf{v_{10}} - -\mathbf{v_{20}}$.

The quantity b is called **impact parameter** and θ **scattering angle**. From conservation laws $v = v'$ and $b = b'$, hence only direction is changed. In general $\theta \equiv \theta\left(\mathbf{v}_{10}, b\right)$, the relationship between b, θ and $\mathbf{v}$ depends on the interaction between the two particles.

In the case of coulombian interaction this is the well known **Rutherford scattering formula**:

$$\tan\frac{\theta}{2} = \frac{b_c}{b} \tag{3.3}$$

where, defined $\mu = \left(m_1 m_2\right) / \left(m_1 + m_2\right)$ the reduced mass of the system,

$$b_c = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mu v^2} \tag{3.4}$$

is the critical impact parameter which corresponds to the impact parameter which produces a deflection of $\pi/2$.

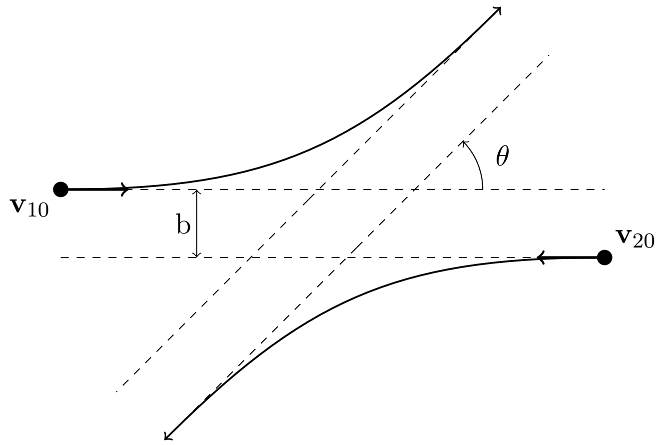


Figure 3.1

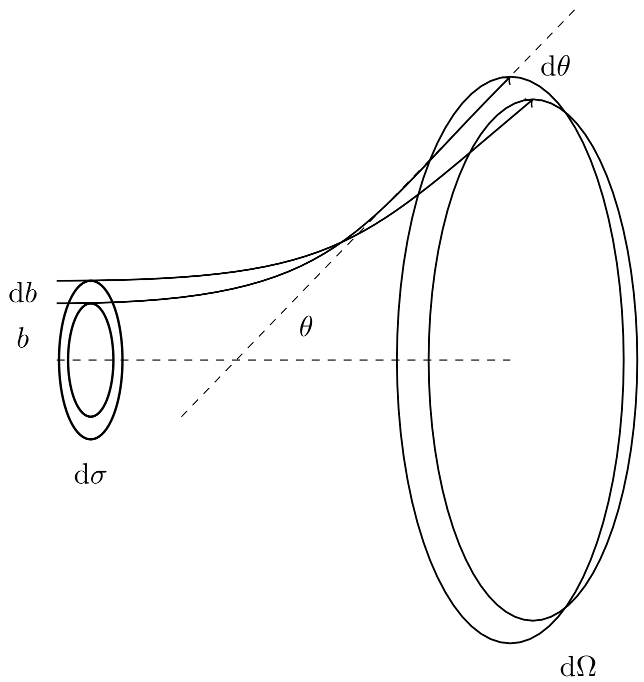


Figure 3.2

In order to evaluate the global effect of any interaction in a complex system, the scattering formula is rewritten in terms of *differential cross section*. The particles deflectd within an angle $[\theta, \theta + \mathrm{d}\theta]$ (corresponding to a solid angle $\mathrm{d}^2\Omega = \sin\theta \mathrm{d}\theta\phi$) have an impact parameter $\in [b, b + \mathrm{d}b]$. We define

$$\begin{cases} \mathrm{d}\sigma = 2\pi b|\mathrm{d}b| \\ \left(\frac{\mathrm{d}\sigma}{\mathrm{d}^2\Omega}\right) = \frac{2\pi b|\mathrm{d}b|}{\sin\theta \mathrm{d}\theta \mathrm{d}\phi}; \end{cases} \tag{3.5}$$

since scattering is independent of ϕ we can integrate $d^2\Omega$ along $d\phi$:

$$d\Omega = \int_0^{2\pi} d\phi \sin\theta d\theta = 2\pi \sin\theta d\theta, \tag{3.6}$$

and finally[1]

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b|db|}{2\pi|\sin(\theta)\,d\theta|} = \frac{b}{\sin(\theta)}\left|\frac{db}{d\theta}\right|. \tag{3.7}$$

From the Rutherford scattering formula we evaluate the *differential cross section for Rutherford scattering* or *coulombian cross section*:

$$\frac{d\sigma}{d\Omega} = \frac{b_c^2}{4\sin(\theta/2)^4} \tag{3.8}$$

3.2 Collisions in a plasma

In a plasma, particles of different species undergo frequent collisions. As one can imagine, the probability for these collisions to occur must be related to the density of particles and to their velocity (hence the plasma temperature); this is qualitatively not different from the case of a gas.

Also interactions among components must have a role in the "thermalization" of the system, i.e. the process that brings the system to its equilibrium.

In a plasma different populations – with different masses and charge states – contribute differently and on disparate timescales the reach of equilibrium. This section is dedicated to a simple calculation that allows to determine the typical thermalization time span in a plasma.

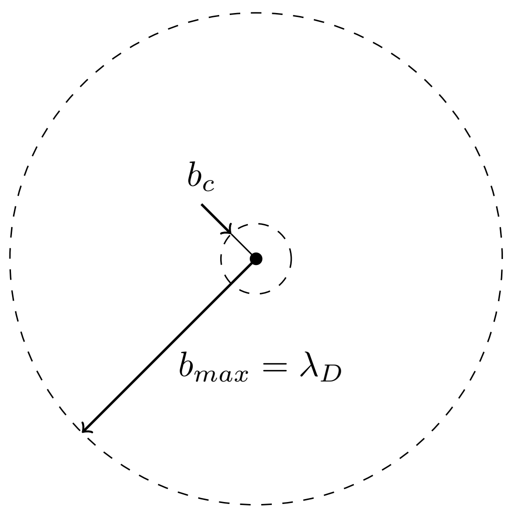


Figure 3.3

In order to quantify the relative importance of different collisions we start by differentiating between *low angle collision*, i.e. $\theta < \pi/2$ and *high angle collision*, i.e. $\theta \geq \pi/2$.

[1] from $\sin^2\theta = 4\tan^2(\theta/2) \;/\; \left(1+\tan^2\theta/2\right)^2$

3.2.1 Small angle collisions

We want here to quantify the timescale a particle needs to exchange energy in collisions where the scattering angle is $< \pi/2$.

As one can notice in (3.8) the differential cross section diverges as $\theta \to 0$ (i.e. for large impact parameters), hence it is necessary to put a higher limit to the impact parameter. The most natural choice is the Debye length, λ_D for one can expect that the scattering center will be effectively screened by electrons at larger distances.

Let's consider a particle propagating with $v = v_x$. We want to define the *mean free path* as the distance travelled by the particle before cumulating a total deviation of $\theta_{total} = \pi/2$.

In the absence of a fluid velocity, there is no reason, in thermal equilibrium, to observe a non void average velocity:

$$\Delta v_x = \sum_i^N (\Delta v_x)_i = 0 \tag{3.9}$$

However, since the searched effect is a net energy transfer, the same discussion does not hold for squared deviation:

$$(\Delta v_x)^2 = \left(\sum_{i=1}^N (\Delta v_x)_i \right)^2 = \sum_{i=1}^N (\Delta v_x)_i^2 + \sum_{j \neq k}^{1..N} (\Delta v_x)_j (\Delta v_x)_k \,. \tag{3.10}$$

Here we neglect off diagonal terms because we expect a low level of correlation between the separate collisions. Given the high quantity of expected collisions, the calculation can be carried out in terms of average deviation $\left\langle (\Delta v_x)_i^2 \right\rangle$, then for the total deviation:

$$\langle \Delta v_x^2 \rangle = N \langle (\Delta v_x)_i^2 \rangle, \tag{3.11}$$

where for a total deflection of $\pi/2$ it should be

$$\Delta v_x^2 = v^2 \tag{3.12}$$

A single small angle collision is obtained by substitution in (3.11) of the Rutherford scattering formula (3.8), using the trigonometric identity $\sin^2(\theta/2) = 4 \tan^2(\theta/2) \,/\, \left(1 + \tan^2(\theta/2)\right)^2$:

$$(\Delta v_x)_i^2 = v^2 \sin^2\theta \quad = \quad v^2 \frac{4\,(b_0/b)^2}{\left[1 + (b_0/b)^2\right]^2} \tag{3.13}$$

$$= \quad v^2 \frac{4\,(b/b_0)^2}{\left[1 + (b/b_0)^2\right]^2} \tag{3.14}$$

where (3.14) has been multiplied by $(b/b_0)^4$.

The average deviation is calculated on the ring surface $2\pi b \mathrm{d}b$ of the possible impact parameters, with inner radius b_c and outer radius λ_D (see Fig.3.2). This gives

$$\begin{aligned}
\langle(\Delta v_x)_i^2\rangle &= \frac{\displaystyle\int_{b_c}^{\lambda_D} 2\pi b v^2 \frac{4\,(b/b_0)^2}{\left[1+(b/b_0)^2\right]^2}\mathrm{d}b}{\displaystyle\int_{b_c}^{\lambda_D} 2\pi b \mathrm{d}b} &(3.15)\\
&= \frac{\displaystyle b_c^2\int_{b_c}^{\lambda_D} 4\pi v^2 \frac{1}{(b/b_c)^2}\,\mathrm{d}\,(b/b_c)^2}{\displaystyle\int_{b_c}^{\lambda_D} 2\pi b\,\mathrm{d}b} &(3.16)\\
&= \frac{8 b_c^2 \pi v^2 \ln(\lambda_D/b_c)}{\displaystyle\int_{b_c}^{\lambda_D} 2\pi b\,\mathrm{d}b} &(3.17)
\end{aligned}$$

The quantity $\Lambda = \ln(\lambda_D/b_c)$ is termed *Coulombian logarithm*; it can be verified that, in a large range of plasma parameters, $\Lambda \sim 10-20$.

Defining now λ_c the mean free path before a total $\pi/2$ deflection is obtained in N collisions, it must hold

$$N = n\lambda_c \int_{b_c}^{\lambda_D} 2\pi b\,\mathrm{d}b \tag{3.18}$$

where n is the plasma density. Finally from (3.11), (3.17) and considering (3.18) one gets:

$$\lambda_c \simeq \frac{1}{8\pi n b_c^2 \Lambda} \tag{3.19}$$

3.2.2 Large angle collisions

In order to compare the effect of large angle to small angle collisions it is enough to calculate $\lambda_{\pi/2}$, the *mean free path* for a single collision with a scattering angle $\theta \geq \pi/2$.

Considering $\Delta v_x = v$ for a $\pi/2$ deflection, for $N_{\pi/2}$ it holds

$$\begin{aligned}
N_{\pi/2} &= n\lambda_{\pi/2}\int_0^{b_c} 2\pi b \mathrm{d}b\\
&= n\lambda_{\pi/2}\pi b_c^2
\end{aligned} \tag{3.20}$$

hence, setting $N_{\pi/2} \equiv 1$, we get

$$\lambda_{\pi/2} = \frac{1}{n\pi b_c^2} \tag{3.21}$$

which compared to λ_c (3.19) gives

$$\lambda_{\pi/2} = 8\Lambda\lambda_c \tag{3.22}$$

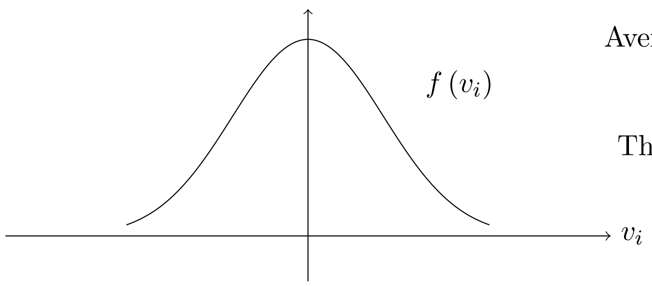
The factor 8Λ is in the order of $\sim 100-200$ which; from (3.22) small angle collisions are far more effective on energy exchange than large angle collisions. Hence these last can safely be neglected. This fact is a consequence of the coulombian force being a long range potential.

3.2.3 Collision time

In a plasma where ions, mass m_i, charge $-Ze$, and electrons, mass m_e, charge e are present, the following possible collisions are considered:

1. $e^- - e^-$
2. e^- – ion
3. ion – ion

In general, at the thermal equilibrium the plasma is isotropic: each velocity component has a zero-centered gaussian distribution (i.e. there is no preferential direction of motion, hence the average is zero!). Given $f(v)$ the velocity distribution.



Average velocity: $\mathbf{u} = \int \mathbf{v} \cdot f(\mathbf{v})\, \mathrm{d}^3 v$

Thermal velocity: $v_t^2 = \dfrac{2k_B T}{m}$

In most of the cases the thermal velocity v_t is very large compared to the fluid velocity u.

- For **electron-electron** collision, $\mu = m_e/2$ and $v = v_{te} = (2k_B T_e/m_e)^{1/2}$, which gives:

$$\tau_{ee} = \frac{\lambda_c}{v_{te}} = \frac{1}{\nu_{ee}} = \left[\frac{n_e e^4 \Lambda}{\sqrt{2}\pi\varepsilon_0^2 m_e^{1/2} (k_B T_e)^{3/2}}\right]^{-1} \tag{3.23}$$

- For **electron-ion collisions**, $\mu \simeq m_e$ and $v \simeq v_e = v_{te} = (2k_B T_e/m_e)^{1/2}$ at the equilibrium, hence the characteristic collision time can be evaluated as:

$$\tau_{ei} = \frac{\lambda_c}{v_{te}} = \frac{1}{\nu_{ei}} = \left[\frac{n_e Z^2 e^4 \Lambda}{4\sqrt{2}\pi\varepsilon_0^2 m_e^{1/2} (k_B T_e)^{3/2}}\right]^{-1} \tag{3.24}$$

$$\tag{3.25}$$

$$\propto \frac{(k_B T_e)^{3/2} m_e^{1/2}}{n_e e^4}. \tag{3.26}$$

- With analogous considerations for **ion-ion collisions**, $\mu = m_i/2$ and $v = v_{ti} = (2k_B T_i/m_i)^{1/2}$, hence:

$$\tau_{ii} = \frac{\lambda_c}{v_{ti}} = \frac{1}{\nu_{ii}} = \left[\frac{n_i Z^4 e^4 \Lambda}{\sqrt{2}\pi\varepsilon_0^2 m_i^{1/2} (k_B T_i)^{3/2}}\right]^{-1} \tag{3.27}$$

$$\tag{3.28}$$

$$\propto \frac{(k_B T_i)^{3/2} m_i^{1/2}}{n_i e^4}. \tag{3.29}$$

The characteristic times τ_{ei} and τ_{ii} are generally very different. Considering for example a $Z = 1$ plasma:

$$\frac{\tau_{ei}}{\tau_{ii}} \simeq \left(\frac{T_e}{T_i}\right)^{3/2} \left(\frac{m_e}{m_i}\right)^{1/2}. \tag{3.30}$$

Apart from a constant factor, $\tau_{ee} \sim \tau_{ei}$, so on a temporal axis, the two dynamics are separated by a factor $\sqrt{m_e/m_i}$ (Fig.3.4).

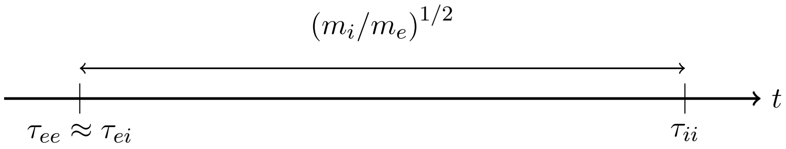


Figure 3.4

The typical time for collisions determine the timescale for system transition to equilibrium.

Let's take a confined plasma evolving to thermal equilibrium due to collisions. The final configuration will be composed by the two components – electrons and ions – following the maxwellian distribution at the same temperature. Due to the large difference between τ_{ee} and τ_{ii}, the electron population is expected to reach the thermal equilibrium faster than the ion population. Moreover the energy exchange between the two populations will also be governed by the mass difference. Considering that in single electron-ion collision the amount of exchanged energy is

$$\frac{\Delta\mathcal{E}}{\mathcal{E}} \approx \frac{m_e}{m_i} \tag{3.31}$$

a number of collisions $\propto m_i/m_e$ will be needed to have the exchanged energy in the order of the mean energy of the two populations.

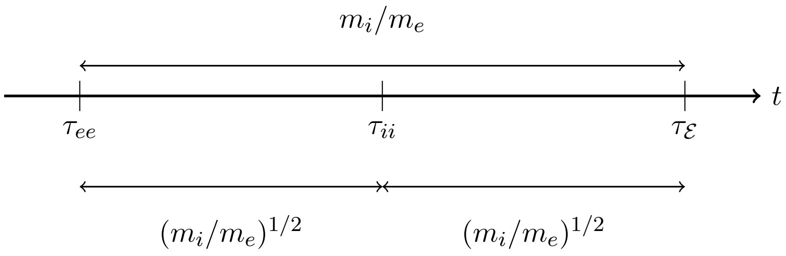


Figure 3.5